

SOME SPECTRAL PROPERTIES OF ROOMS AND PASSAGES DOMAINS AND THEIR SKELETONS

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Dedicated to Fritz Gesztesy on the occasion of his sixtieth birthday.

ABSTRACT. In this paper we investigate spectral properties of Laplacians on Rooms and Passages domains. In the first part, we use Dirichlet-Neumann bracketing techniques to show that for the Neumann Laplacian in certain Rooms and Passages domains the second term of the asymptotic expansion of the counting function is of order $\sqrt{\lambda}$. For the Dirichlet Laplacian our methods only give an upper estimate of the form $\sqrt{\lambda}$. In the second part of the paper, we consider the relationship between Neumann Laplacians on Rooms and Passages domains and Sturm-Liouville operators on the skeleton.

1. INTRODUCTION

Let $-\Delta_{N,\Omega}$ denote the Neumann Laplacian on a bounded open subset Ω of \mathbb{R}^n , $n \geq 2$, and let $E : H^1(\Omega) \rightarrow L^2(\Omega)$ be the canonical embedding, where $H^1(\Omega)$ is the standard Sobolov space. Then $(-\Delta_{N,\Omega} + 1)^{-1} = EE^*$, and $-\Delta_{N,\Omega}$ has a compact resolvent (and thus a discrete spectrum) if and only if E is compact. The compactness, or otherwise, of E is determined by the nature of the boundary $\partial\Omega$ of Ω . In [1] it was shown that E being compact is equivalent to a *generalised extension property*, which is that *there exists a function space $\Lambda(\mathbb{R}^n)$ on \mathbb{R}^n which is compactly embedded in $L^2(B)$ for any ball $B \subset \mathbb{R}^n$, and is such that there is a continuous extension $\mathcal{E} : H^1(\Omega) \rightarrow \Lambda(\mathbb{R}^n)$.* In general $\Lambda(\mathbb{R}^n)$ is a space of arbitrary smoothness. If $\partial\Omega \in Lip_\alpha$, $0 < \alpha < 1$, i.e., $\partial\Omega$ coincides with the subgraph of a Lip_α function in a neighbourhood of each point, then $\Lambda(\mathbb{R}^n) = H^\alpha(\mathbb{R}^n)$, the Sobolev space of order α , so that, in this case, there is a reduction of smoothness in going from Ω to \mathbb{R}^n . For domains with singular boundaries, like the “Rooms and Passages” domain defined in Section 2, the compactness of the embedding E can be taken as a measure of the smoothness of the boundary, which is otherwise difficult to describe. When E is not compact, [7] shows that for any closed subset S of the non-negative real numbers, there exists a modified “Rooms and Passages” or a “Comb” domain such that S equals the essential spectrum of $-\Delta_{N,\Omega}$.

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Suppose that $-\Delta_{N,\Omega}$ has a discrete spectrum and denote the number of its eigenvalues less than λ by $N_N(\lambda)$. The problem of determining the asymptotic behaviour of $N_N(\lambda)$ as $\lambda \rightarrow \infty$ has a long history. If the boundary $\partial\Omega$ of Ω is sufficiently smooth, it has long been known that one has the asymptotic formula

$$(1.1) \quad N_N(\lambda) = \omega_n(2\pi)^{-n}|\Omega|\lambda^{n/2} + o(\lambda^{n/2}),$$

where ω_n is the volume of the unit ball in \mathbb{R}^n and $|\Omega|$ is the volume of the domain Ω . For general Ω the Weyl term $\omega_n(2\pi)^{-n}|\Omega|\lambda^{n/2}$ dominates, and in the error $R(\lambda) := N_N(\lambda) - \omega_n(2\pi)^{-n}|\Omega|\lambda^{n/2}$, it is the boundary $\partial\Omega$, rather than any other topological or geometrical feature of Ω which is dominant. For instance, in [9], it is shown that if $\partial\Omega \in Lip_\alpha$, $0 < \alpha < 1$, the remainder $R(\lambda) = O(\lambda^{(n-1)/2\alpha})$ and this is order sharp. Here and in the following, the $O(f(\lambda))$ - and $o(f(\lambda))$ -notation is to be understood in the limit $\lambda \rightarrow \infty$. Moreover, in [4], a domain of von-Koch snowflake type Ω is exhibited for which $R(\lambda) \asymp \lambda^{d_o}$, where d_o denotes the outer Minkowski dimension of the boundary and $A(\lambda) \asymp B(\lambda)$ means that $|A(\lambda)/B(\lambda)|$ is bounded above and below by positive constants as $\lambda \rightarrow \infty$. Specific information about the error is hard to come by. For general domains one of the few available techniques is the Courant-Weyl variational method involving “Dirichlet-Neumann bracketing”. This effectively reduces the problem to estimating the corresponding counting functions $N_D(\lambda)$ and $N_N(\lambda)$ for Dirichlet and Neumann Laplacians on a set of cubes which cover Ω . A variant of this technique is used in [9] in which cubes are replaced by other relatively simple sets for which the Dirichlet and Neumann Laplacian counting functions can be estimated from above and below. The first part of this paper is a contribution to the study of how the error term $R(\lambda)$ depends on the boundary $\partial\Omega$. We look in detail at the much studied “Rooms and Passages” domain, in which the Weyl formula (1.1) holds, being particularly concerned with upper and lower bounds for $R(\lambda)\lambda^{-1/2}$.

In [5] it was shown that for a rather restricted class of domains Ω , (which does not include Rooms and Passages) $-\Delta_{N,\Omega}$ has a compact resolvent if and only if a Sturm-Liouville operator defined on the *skeleton* of Ω has a compact resolvent. Recall that the *skeleton* of an open set Ω is the complement of the set of points \mathbf{x} in Ω for which there exists a unique point \mathbf{y} on $\partial\Omega$ such that $|\mathbf{y} - \mathbf{x}|$ is equal to the distance of \mathbf{x} from $\mathbb{R}^n \setminus \Omega$. The result in [5] was motivated by Theorem 3.3 in [2] in which Ω is a horn, whose skeleton is a half-line. In the second part of the paper (Sections 4 and 5) we investigate this problem for general Rooms and Passages domains.

2. ROOMS & PASSAGES DOMAINS

We consider a Rooms and Passages (R&P for short) domain Ω defined as the union of square rooms R_i (i odd) of size $h_i \times h_i$ joined by rectangular

passages P_i (i even) of size $h_i \times \delta_i$ with $\delta_i < \min\{h_{i-1}, h_{i+1}\}$. We assume $h_i \rightarrow 0$ as $i \rightarrow \infty$. See Figure 1.

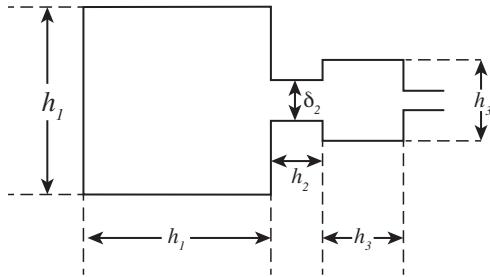


FIGURE 1. The start of a general Rooms and Passages domain.

In this section and the next, we further restrict the R&P domains under consideration by assuming that $h_i = C^i$ and $\delta_i = kC^{i\alpha}$, where $C < 1$, $\alpha > 1$ and k are constants, with $k < C^{3-2\alpha}$. This guarantees that each passage is narrower than both adjacent rooms. It follows from the general result in [3], Example 6.1.1, that the embedding $E : H^1(\Omega) \rightarrow L^2(\Omega)$ is compact if and only if $\alpha < 3$.

Proposition 2.1. *For $\alpha > 3$ we have $0 \in \sigma_{ess}(-\Delta_{N,\Omega})$ where σ_{ess} denotes the essential spectrum of the operator.*

Proof. We give a simple singular sequence proof. By Ω_j we denote the domain consisting of the first j rooms and passages while $T_j = \Omega \setminus \Omega_j$ denotes the ‘tail’. Choose a sequence of C^∞ functions φ_j which depend only on the x -variable such that $\text{supp } \varphi_j \subseteq \Omega_{4j} \setminus \Omega_{2j-1}$,

$$(2.1) \quad \varphi_j = \begin{cases} 1 & \text{on } \Omega_{4j-1} \setminus \Omega_{2j}, \\ 0 & \text{on } \Omega_{4j}^c \cup \Omega_{2j-1}, \end{cases} \quad \text{and} \quad |\varphi'_j| = \begin{cases} O(C^{-2j}) & \text{on } \Omega_{2j} \setminus \Omega_{2j-1}, \\ O(C^{-4j}) & \text{on } \Omega_{4j} \setminus \Omega_{4j-1}. \end{cases}$$

In other words, the function φ_j is initially zero, increases to 1 in the j -th passage and falls back to zero in the $2j$ -th passage. The support of the sequence of the φ_j ‘disappears’ into the tail and each φ_j satisfies the Neumann boundary condition on $\partial\Omega$. The singular sequence we wish to consider is given by

$$(2.2) \quad f_j(x, y) = \frac{1}{\|\varphi_j\|_{L^2(\Omega)}} \varphi_j(x) \quad \text{for } (x, y) \in \Omega.$$

Obviously, $\|f_j\|_{L^2(\Omega)} = 1$, while f_j weakly converges to 0 in $L^2(\Omega)$ as $j \rightarrow \infty$.

We have the following:

$$(2.3) \quad \|\varphi_j\|_{L^2(\Omega)} \asymp (|\Omega_{4j}| - |\Omega_{2j}|)^{1/2} \quad \text{as } j \rightarrow \infty$$

and

$$\begin{aligned}
 |\Omega_{4j}| &= \sum_{i=1, i \text{ odd}}^{4j-1} C^{2i} + \sum_{i=1, i \text{ even}}^{4j} kC^{i(1+\alpha)} \\
 (2.4) \quad &= C^2 \frac{1 - C^{8j}}{1 - C^4} + kC^{2(1+\alpha)} \frac{1 - C^{4j(1+\alpha)}}{1 - C^{2(1+\alpha)}}.
 \end{aligned}$$

Therefore,

$$|\Omega_{4j}| - |\Omega_{2j}| = C^2 \frac{C^{4j} - C^{8j}}{1 - C^4} + kC^{2(1+\alpha)} \frac{C^{2j(1+\alpha)} - C^{4j(1+\alpha)}}{1 - C^{2(1+\alpha)}} \asymp C^{4j} \text{ as } j \rightarrow \infty,$$

and for large j , $\|\varphi_j\|_{L^2(\Omega)}$ behaves like C^{2j} .

As φ'_j is supported only on two passages we get from (2.1),

$$\begin{aligned}
 (2.5) \quad &\|\varphi'_j\|_{L^2(\Omega)}^2 = O(C^{-8j} k C^{4j(1+\alpha)} + C^{-4j} k C^{2j(1+\alpha)}) = O(C^{-2j(1-\alpha)}) \text{ as } j \rightarrow \infty.
 \end{aligned}$$

Hence, as $j \rightarrow \infty$,

$$\begin{aligned}
 (2.6) \quad &\|f'_j\|_{L^2(\Omega)} = \frac{1}{\|\varphi_j\|_{L^2(\Omega)}} \|\varphi'_j\|_{L^2(\Omega)} = O(C^{-j(1-\alpha)} C^{-2j}) = O(C^{-j(3-\alpha)}) \rightarrow 0
 \end{aligned}$$

if $\alpha > 3$. This implies that $(-\Delta)^{1/2} f_j \rightarrow 0$ as $j \rightarrow \infty$ although $\|f_j\|_{L^2(\Omega)} = 1$.

Therefore, we have $0 \in \sigma_{ess}((-\Delta_{N,\Omega})^{1/2})$, which implies $0 \in \sigma_{ess}(-\Delta_{N,\Omega})$ when $\alpha > 3$. \square

3. EIGENVALUE ASYMPTOTICS OF THE DIRICHLET AND NEUMANN LAPLACIANS

The special class of R&P domains introduced in the previous section will be considered. Let $\alpha < 3$ and denote by $N_D(\lambda)$, $N_N(\lambda)$, respectively, the counting functions of the Dirichlet and Neumann Laplacians on the R&P domain Ω . To determine their asymptotic behaviour, we shall apply the Dirichlet-Neumann bracketing technique.

We begin this analysis of the spectrum of the Laplacian on an R&P domain Ω by discussing the contribution from the tail. As before, let $\Omega = \Omega_{2M} \cup T_{2M}$ where Ω_{2M} consists of the first $2M$ rooms and passages and T_{2M} denotes the tail. On applying Theorems 4.6 and 5.1 to Example 6.1.1 in [3], it follows¹ that the optimal constant $K(T_{2M})$ in the Poincaré inequality

$$\|f - f_{T_{2M}}\|_{L^2(T_{2M})} \leq K(T_{2M}) \|\nabla f\|_{L^2(T_{2M})}, \quad (f \in H^1(T_{2M})),$$

satisfies

$$K(T_{2M}) \leq c C^{(3-\alpha)M}$$

¹By Theorem 4.6, the bound on $K(T_{2M})$ involves a quantity $c(J_1)$, which according to Theorem 5.1 is bounded by $\sqrt{N_a}$. Noting that their constant C corresponds to our C^{-1} , Theorem 6.1 and Example 6.1.1 with $k = 2M$ then give $N_a \asymp C^{2M(3-\alpha)}$.

where c is a positive constant.

It follows that the Neumann Laplacian on T_{2M} will only contribute the trivial eigenvalue 0 if

$$(3.1) \quad \lambda < (1/c)^2 C^{2(\alpha-3)M}, \quad \text{i.e.} \quad M > \frac{\log(c^2\lambda)}{2(3-\alpha)\log C^{-1}}.$$

Since $\alpha > 1$, the tail after M passages has area

$$|T_{2M}| = \frac{C^{2+4M}}{1-C^4} + \frac{kC^{2(1+\alpha)}C^{2M(1+\alpha)}}{1-C^{2(1+\alpha)}} \asymp C^{4M} \text{ as } M \rightarrow \infty.$$

It follows that for M as in (3.1)

$$|T_{2M}| = O(\lambda^{-2/(3-\alpha)})$$

and so

$$|\Omega| = |\Omega_{2M}| + o(\sqrt{\lambda}).$$

Now, for M satisfying (3.1), the counting function for the Neumann-Laplacian on Ω differs by at most 1 from that on Ω_{2M} . It follows that

$$(3.2) \quad R(\lambda) - R_{2M}(\lambda) = o(\sqrt{\lambda}),$$

where $R(\lambda)$ and $R_{2M}(\lambda)$ denote the error term for the problem on Ω and Ω_{2M} , respectively.

As we have that the $(n+1)$ -th Neumann eigenvalue is a lower bound for the n -th Dirichlet eigenvalue (see [6]), for λ satisfying (3.1), the tail cannot contribute any Dirichlet eigenvalue, so the same reasoning as for the Neumann case implies that (3.2) also holds in the Dirichlet case.

3.1. Asymptotics for $N_N(\lambda)$. Our strategy here is to partition the domain and use the Dirichlet Neumann bracketing technique to obtain the required estimates. In order to obtain these estimates we first obtain a lower bound for the number of eigenvalues of the Neumann-Laplacian, we partition the rooms into five sections imposing Neumann boundary conditions on the boundary of Ω and Dirichlet boundary conditions on all artificially introduced internal boundaries (see Figure 2). An upper bound is obtained by only introducing an artificial boundary to separate the room from the neighbouring passages and imposing Neumann conditions on all the boundaries (see Figure 3). This is a simple consequence of the variational principle.

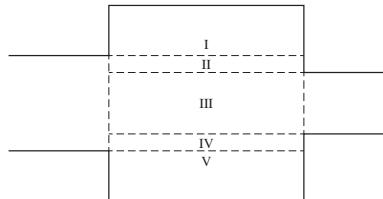


FIGURE 2. Artificial boundaries and sub-regions of a room for the lower bound: Neumann conditions on the dotted lines.

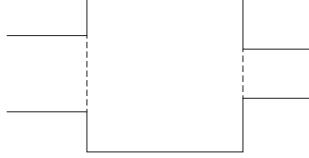


FIGURE 3. Artificial boundaries of a room for the upper bound: Dirichlet conditions on the dotted lines.

We first consider the one-dimensional problem on $[-a, a]$ and have the following eigenfunctions and eigenvalues:

(a) Dirichlet conditions at both end points:

$$\psi_m(x) = c \sin \frac{m\pi(x+a)}{2a}, \quad \lambda_m = \frac{m^2\pi^2}{4a^2}, \quad m \geq 1.$$

(b) Dirichlet conditions at $-a$ and Neumann at a :

$$\psi_m(x) = c \sin \frac{(2m+1)\pi(x+a)}{4a}, \quad \lambda_m = \frac{(2m+1)^2\pi^2}{16a^2}, \quad m \geq 0.$$

(c) Neumann conditions at both end points:

$$\psi_m(x) = c \cos \frac{m\pi(x+a)}{2a}, \quad \lambda_m = \frac{m^2\pi^2}{4a^2}, \quad m \geq 0.$$

For the lower estimate for a room, this leads to the following.

(1) In regions I and V, we have the set of eigenvalues

$$\left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{(2n+1)^2\pi^2}{16b^2} : m, n \geq 0 \right\} \quad \text{with } a = \frac{C^j}{2}, b = \frac{C^j - kC^{\alpha(j-1)}}{4}.$$

(2) In regions II and IV, we have the set of eigenvalues

$$\left\{ \lambda_{m,n} = \frac{(2m+1)^2\pi^2}{16a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1 \right\} \\ \text{with } a = \frac{C^j}{2}, b = \frac{k}{4} (C^{\alpha(j-1)} - C^{\alpha(j+1)}).$$

(3) In region III, we have the set of eigenvalues

$$\left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m, n \geq 1 \right\} \quad \text{with } a = \frac{C^j}{2}, b = k \frac{C^{\alpha(j+1)}}{2}.$$

We now need to count the integer lattice points in the first quadrant satisfying $\lambda_{m,n} \leq \lambda$. By [8], the number of integer lattice points in a plane region $X \cdot R$, where X is a real scaling parameter, is given by

$$\mathcal{N}(X) = AX^2 + o(X^{\frac{131}{208}+\varepsilon}) \text{ as } X \rightarrow \infty$$

for any $\varepsilon > 0$, where A is the area of the region R . This gives us the following:

(1) In regions I and V, we have an ellipse with half axes $\frac{2a\sqrt{\lambda}}{\pi}$ and $\frac{2b\sqrt{\lambda}}{\pi}$ which is shifted by $-1/2$ in the y -direction. As an upper estimate for the area below the x -axis we use the area of the rectangle which is subtracted from the area of the quarter ellipse (the error from this can easily be seen to be $o(\sqrt{\lambda})$). We then add the contributions along the two axes to get

$$\begin{aligned} \text{card } & \left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{(2n+1)^2\pi^2}{16b^2} : m, n \geq 0, \lambda_{m,n} \leq \lambda \right\} \\ & \geq \frac{\pi}{4} \frac{2a\sqrt{\lambda}}{\pi} \frac{2b\sqrt{\lambda}}{\pi} - \frac{1}{2} \frac{2a\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}) + \frac{2a\sqrt{\lambda}}{\pi} + \frac{2b\sqrt{\lambda}}{\pi} \\ & = \frac{ab}{\pi} \lambda + \frac{(2b+a)\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}). \end{aligned}$$

Here, $\text{card } A$ denotes the cardinality of the finite set A .

(2) In regions II and IV, we again have an ellipse with half axes $\frac{2a\sqrt{\lambda}}{\pi}$ and $\frac{2b\sqrt{\lambda}}{\pi}$ which this time is shifted by $-1/2$ in the x -direction. As an upper estimate for the area left of the y -axis we use the area of the rectangle which is subtracted from the total area. We then add the contributions along the y -axis to get

$$\begin{aligned} \text{card } & \left\{ \lambda_{m,n} = \frac{(2m+1)^2\pi^2}{16a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1, \lambda_{m,n} \leq \lambda \right\} \\ & \geq \frac{\pi}{4} \frac{2a\sqrt{\lambda}}{\pi} \frac{2b\sqrt{\lambda}}{\pi} - \frac{1}{2} \frac{2b\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}) + \frac{2b\sqrt{\lambda}}{\pi} \\ & = \frac{ab}{\pi} \lambda + \frac{b\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}). \end{aligned}$$

(3) In region III, we count the lattice points in an ellipse with half axes $\frac{2a\sqrt{\lambda}}{\pi}$ and $\frac{2b\sqrt{\lambda}}{\pi}$ to give

$$\begin{aligned} \text{card } & \left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m, n \geq 1, \lambda_{m,n} \leq \lambda \right\} \\ & \geq \frac{\pi}{4} \frac{2a\sqrt{\lambda}}{\pi} \frac{2b\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}) = \frac{ab}{\pi} \lambda - o(\sqrt{\lambda}). \end{aligned}$$

Collecting these results, we obtain a lower estimate for the contribution of the j -th room to the counting function of the form

$$N_j \geq \frac{C^{2j}}{4\pi} \lambda + \left[2C^j - \frac{k}{2} (C^{\alpha(j+1)} + C^{\alpha(j-1)}) \right] \frac{\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}).$$

For the upper estimate of the counting function, we simply need to consider the eigenvalues

$$\left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m, n \geq 0 \right\}$$

of the Neumann-Laplacian on the square, where $a = b = \frac{C^j}{2}$. In order to count the integer lattice points, we take the area of the ellipse with half axes $\frac{2a\sqrt{\lambda}}{\pi}$ and $\frac{2b\sqrt{\lambda}}{\pi}$ and add the additional points along the x - and y -axes:

$$\begin{aligned} & \text{card } \left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m, n \geq 0, \lambda_{m,n} \leq \lambda \right\} \\ & \leq \frac{\pi}{4} \frac{2a\sqrt{\lambda}}{\pi} \frac{2b\sqrt{\lambda}}{\pi} + \frac{2a\sqrt{\lambda}}{\pi} + \frac{2b\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda}) \\ & = \frac{ab}{\pi} \lambda + \frac{2a\sqrt{\lambda}}{\pi} + \frac{2b\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda}). \end{aligned}$$

As an upper estimate for the contribution of the j -th room, we therefore get

$$N_j \leq \frac{C^{2j}}{4\pi} \lambda + \frac{2C^j}{\pi} \sqrt{\lambda} + o(\sqrt{\lambda}).$$

The calculations involving the first room are a little different and we only use three partitions (see Figure 4).

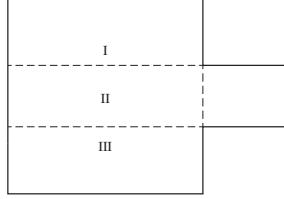


FIGURE 4. Subdivisions of the first room for the lower bound: Neumann conditions on the dotted lines.

(1) In regions I and III, we have the set of eigenvalues

$$\left\{ \frac{m^2\pi^2}{4a^2} + \frac{(2n+1)^2\pi^2}{16b^2} : m, n \geq 0 \right\} \quad \text{with } a = \frac{C}{2}, b = \frac{C - kC^{2\alpha}}{4}.$$

(2) In region II, we have the set of eigenvalues

$$\left\{ \frac{(2m+1)^2\pi^2}{16a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1 \right\} \quad \text{with } a = \frac{C}{2}, b = \frac{kC^{2\alpha}}{2}.$$

Proceeding as for the other rooms we again need to count integer lattice points in the first quadrant.

- In regions I and III, we have

$$\begin{aligned} & \text{card } \left\{ \lambda_{m,n} = \frac{m^2\pi^2}{4a^2} + \frac{(2n+1)^2\pi^2}{16b^2} : m, n \geq 0, \lambda_{m,n} \leq \lambda \right\} \\ & \geq \frac{ab}{\pi} \lambda + \frac{(2b+a)\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}). \end{aligned}$$

- In region II, we have

$$\begin{aligned} \text{card } & \left\{ \lambda_{m,n} = \frac{(2m+1)^2\pi^2}{16a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1, \lambda_{m,n} \leq \lambda \right\} \\ & \geq \frac{ab}{\pi}\lambda + \frac{b\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}). \end{aligned}$$

Combining these we get a lower estimate for the contribution of the first room

$$N_1 \geq \frac{\lambda C^2}{4\pi} + \left(2C - \frac{k}{2}C^{2\alpha} \right) \frac{\sqrt{\lambda}}{\pi} - o(\sqrt{\lambda}).$$

We next sum over the rooms (omitting the $o(\sqrt{\lambda})$ -term for convenience). Denoting the volume of the first M rooms by $V(R_M)$, we see

$$\begin{aligned} \sum_{j=1, j \text{ odd}}^{2M} N_j & \geq \sum_{j=1, j \text{ odd}}^{2M} \frac{\lambda C^{2j}}{4\pi} \\ & + \left(2 \sum_{j=1, j \text{ odd}}^{2M} C^j - \frac{k}{2} \left(C^{2\alpha} + \sum_{j=3, j \text{ odd}}^{2M} C^{\alpha(j-1)} (1 + C^{2\alpha}) \right) \right) \frac{\sqrt{\lambda}}{\pi} \\ & = \frac{V(R_M)}{4\pi} \lambda + \left(2C \frac{1 - C^{2M}}{1 - C^2} - \frac{k}{2} \cdot \frac{2C^{2\alpha} - C^{2\alpha M} - C^{2\alpha(M+1)}}{1 - C^{2\alpha}} \right) \frac{\sqrt{\lambda}}{\pi}. \end{aligned}$$

Also,

$$\sum_{j=1, j \text{ odd}}^{2M} N_j \leq \sum_{j=1, j \text{ odd}}^{2M} \left(\frac{\lambda C^{2j}}{4\pi} + \frac{2\sqrt{\lambda}C^j}{\pi} \right) = \frac{V(R_M)}{4\pi} \lambda + \frac{2\sqrt{\lambda}}{\pi C} \left(\frac{C^2 - C^{2(M+1)}}{1 - C^2} \right).$$

We now proceed to estimate the counting function in a passage. Here,

$$a = \frac{C^j}{2}, \quad b = \frac{kC^{\alpha j}}{2} \quad \text{and} \quad \lambda_{m,n} = \frac{\pi^2 m^2}{C^{2j}} + \frac{\pi^2 n^2}{C^{2\alpha j} k^2}, \quad m \geq 1, n \geq 0.$$

Then, since the lattice point counting estimate is on the ellipse with semi axes $\frac{2a\sqrt{\lambda}}{\pi}$ and $\frac{2b\sqrt{\lambda}}{\pi}$, we have

$$N_j \geq \frac{\lambda}{\pi} ab - o(\sqrt{\lambda}) + \frac{2a\sqrt{\lambda}}{\pi} = \frac{kC^{(1+\alpha)j}}{4\pi} \lambda + \frac{C^j}{\pi} \sqrt{\lambda} - o(\sqrt{\lambda}).$$

For the upper bound we count the eigenvalues

$$\lambda_{m,n} = \frac{\pi^2 m^2}{C^{2j}} + \frac{\pi^2 n^2}{C^{2\alpha j} k^2}, \quad m \geq 0, n \geq 0.$$

Thus

$$N_j \leq \frac{\lambda}{\pi} ab + o(\sqrt{\lambda}) + \frac{2a\sqrt{\lambda}}{\pi} + \frac{2b\sqrt{\lambda}}{\pi} = \frac{kC^{(1+\alpha)j}}{4\pi} \lambda + \left(\frac{kC^{\alpha j}}{\pi} + \frac{C^j}{\pi} \right) \sqrt{\lambda} + o(\sqrt{\lambda}).$$

Summing, we obtain bounds for the counting function in the passages

$$\begin{aligned} \sum_{j=1, even}^{2M} N_j &\geq \sum_{j=1, even}^{2M} \frac{kC^{(1+\alpha)j}}{4\pi} \lambda + \frac{C^j}{\pi} \sqrt{\lambda} = \frac{V(P_M)}{4\pi} \lambda + \sum_{i=1}^M \frac{C^{2i}}{\pi} \sqrt{\lambda} \\ &= \frac{V(P_M)}{4\pi} \lambda + \frac{\sqrt{\lambda}}{\pi} \frac{C^2 - C^{2(M+1)}}{1 - C^2}, \end{aligned}$$

where $V(P_M)$ denotes the volume of the first M passages. We have the upper bound

$$\begin{aligned} \sum_{j=1, even}^{2M} N_j &\leq \frac{V(P_M)}{4\pi} \lambda + \sum_{j=1, j even}^{2M} \left(\frac{kC^{\alpha j}}{\pi} + \frac{C^j}{\pi} \right) \sqrt{\lambda} \\ &= \frac{V(P_M)}{4\pi} \lambda + \frac{\sqrt{\lambda}}{\pi} \left(k \sum_{i=1}^M C^{2\alpha i} + \sum_{i=1}^M C^{2i} \right) \\ &= \frac{V(P_M)}{4\pi} \lambda + \left(k \frac{C^{2\alpha} - C^{2\alpha(1+M)}}{1 - C^{2\alpha}} + \frac{C^2 - C^{2(1+M)}}{1 - C^2} \right) \frac{\sqrt{\lambda}}{\pi}. \end{aligned}$$

We now are in a position to estimate the bounds of the counting function for the domain Ω_{2M} . Summing the contributions from the rooms and the passages, we get the lower estimate

$$\frac{|\Omega_{2M}|}{4\pi} \lambda + \left(\frac{2C + C^2 - 2C^{2M+1} - C^{2M+2}}{1 - C^2} - \frac{k}{2} \frac{2C^{2\alpha} - C^{2\alpha M} - C^{2\alpha(M+1)}}{1 - C^{2\alpha}} \right) \frac{\sqrt{\lambda}}{\pi}$$

with the upper estimate given by

$$\frac{|\Omega_{2M}|}{4\pi} \lambda + \left(\frac{2C + C^2 - 2C^{2M+1} - C^{2M+2}}{1 - C^2} + k \frac{C^{2\alpha} - C^{2\alpha(1+M)}}{1 - C^{2\alpha}} \right) \frac{\sqrt{\lambda}}{\pi}.$$

As $M \rightarrow \infty$, since there is no contribution from the tail and $C < 1$ we get

$$\frac{\sqrt{\lambda}}{\pi} \left(\frac{2C + C^2}{1 - C^2} - k \frac{C^{2\alpha}}{1 - C^{2\alpha}} \right) \leq N_N(\lambda) - \frac{|\Omega|}{4\pi} \lambda \leq \frac{\sqrt{\lambda}}{\pi} \left(\frac{2C + C^2}{1 - C^2} + \frac{kC^{2\alpha}}{1 - C^{2\alpha}} \right).$$

These yield the following result.

Theorem 3.1. *As $\lambda \rightarrow \infty$,*

$$C_1 \frac{\sqrt{\lambda}}{\pi} \leq N_N(\lambda) - \frac{|\Omega|}{4\pi} \lambda \leq C_2 \frac{\sqrt{\lambda}}{\pi}$$

where

$$C_1 = \frac{2C + C^2}{1 - C^2} - k \frac{C^{2\alpha}}{1 - C^{2\alpha}} > 0, \quad C_2 = \frac{2C + C^2}{1 - C^2} + \frac{kC^{2\alpha}}{1 - C^{2\alpha}}$$

and

$$C_2 - C_1 = 2k \frac{C^{2\alpha}}{1 - C^{2\alpha}} \rightarrow 0 \quad \text{as } k \rightarrow 0.$$

Therefore, the error is precisely of order $\sqrt{\lambda}$ and in the limit $k \rightarrow 0$, we obtain the precise constant.

3.2. Asymptotics for $N_D(\lambda)$. We now estimate bounds for the counting function of the Dirichlet-Laplacian. In order to do this we again look separately at the rooms and the passages. The lower bound is obtained by choosing Dirichlet conditions on all the boundaries. The set of eigenvalues is given by

$$\left\{ \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m, n \geq 1 \right\} \quad \text{with } a = \frac{C^j}{2}, b = \frac{1}{2} \begin{cases} C^j & \text{for rooms,} \\ kC^{\alpha j} & \text{for passages.} \end{cases}$$

Therefore, the contribution to the lower bound from each room or passage is $\frac{ab}{\pi}\lambda - o(\sqrt{\lambda})$. Adding all contributions, we get the lower bound

$$N_D(\lambda) \geq \frac{|\Omega|}{4\pi}\lambda + o(\sqrt{\lambda}).$$

To get an upper bound in the first room, we choose Dirichlet conditions on three sides of the boundary and Neumann conditions on the right side. This gives us the set of eigenvalues

$$\left\{ \frac{(2m+1)^2\pi^2}{16a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1 \right\} \quad \text{with } a = b = \frac{C}{2}.$$

Counting lattice points gives $\frac{ab}{\pi}\lambda + \frac{b\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda})$, so as an upper estimate for the contribution of the first room, we get

$$\frac{\lambda C^2}{4\pi} + \frac{C}{2} \frac{\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda}).$$

For the remaining rooms and passages we choose Dirichlet conditions on the horizontal boundaries and Neumann conditions on the vertical ones. This gives us the set of eigenvalues

$$\left\{ \frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} : m \geq 0, n \geq 1 \right\}.$$

Again counting lattice points gives $\frac{ab}{\pi}\lambda + \frac{2b\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda})$. Summing all contributions yields as the upper estimate (omitting the $o(\sqrt{\lambda})$ -term).

$$\begin{aligned} & \frac{|\Omega|}{4\pi}\lambda + \left(\frac{C}{2} + \sum_{j=3, j \text{ odd}}^{2M} C^j + k \sum_{j=1, j \text{ even}}^{2M} C^{j\alpha} \right) \frac{\sqrt{\lambda}}{\pi} \\ &= \frac{|\Omega|}{4\pi}\lambda + \left(C \frac{1 - C^{2M}}{1 - C^2} - \frac{C}{2} + kC^{2\alpha} \frac{1 - C^{2\alpha M}}{1 - C^{2\alpha}} \right) \frac{\sqrt{\lambda}}{\pi}. \end{aligned}$$

Therefore, letting $M \rightarrow \infty$, we find the following result.

Theorem 3.2. *As $\lambda \rightarrow \infty$*

$$o(\sqrt{\lambda}) \leq N_D(\lambda) - \frac{|\Omega|}{4\pi}\lambda \leq \left(\frac{C(C^2 + 1)}{2(1 - C^2)} + \frac{k}{C^{-2\alpha} - 1} \right) \frac{\sqrt{\lambda}}{\pi} + o(\sqrt{\lambda}).$$

We remark that the lower bound is given by having Dirichlet boundary conditions everywhere which does not give a $\sqrt{\lambda}$ -term. Therefore, we can only get an $o(\sqrt{\lambda})$ error estimate and not determine the sign of the $\sqrt{\lambda}$ -term.

4. A RELATED PROBLEM ON THE SKELETON

We define $\mathbf{y} \in \partial\Omega$ to be a near point of $\mathbf{x} \in \Omega$ if $|\mathbf{x} - \mathbf{y}| = \text{dist}(\mathbf{x}, \partial\Omega)$, the distance of \mathbf{x} to the boundary of Ω . Therefore, denoting by $\mathcal{N}(\mathbf{x})$ the set of near points of \mathbf{x} , the skeleton Γ of Ω is the set

$$\mathcal{S}(\Omega) := \{\mathbf{x} \in \Omega : \text{card } \mathcal{N}(\mathbf{x}) > 1\}.$$

It follows that the skeleton of the R&P domain Ω is the union of a sequence of line segments and parabolic arcs, $\Gamma = \{e_j\}_{j \in \mathbb{N}}$ say, connecting points in Ω ; see Figure 5. For any $\mathbf{x} \in \Omega$, there exists $\mathbf{t} \in \Gamma$ such that \mathbf{x} lies on one of two line segments $C_+(\mathbf{t}), C_-(\mathbf{t})$ connecting \mathbf{t} to its 2 near points $\mathbf{y}_+, \mathbf{y}_-$: set $\tau : \Omega \rightarrow \Gamma, \mathbf{x} \rightarrow \mathbf{t}$. If $\tau(\mathbf{x}) = \mathbf{t} \in e \in \Gamma$, we may therefore define the following co-ordinate system on $\tau^{-1}(e)$:

$$(4.1) \quad \mathbf{x} = \mathbf{x}(\sigma, s), \quad \tau(\mathbf{x}) = \mathbf{t} = \mathbf{t}(\sigma), \quad s \in (-l(\sigma), l(\sigma)),$$

where σ denotes arc length along e , s is the distance from \mathbf{t} to \mathbf{x} along $C_+(\mathbf{t}(\sigma))$ or $C_-(\mathbf{t}(\sigma))$, $l(\sigma)$ is the length of $C_+(\mathbf{t}(\sigma))$ and $C_-(\mathbf{t}(\sigma))$ and we set $0 \leq s \leq l(\mathbf{t}(\sigma))$ along $C_+(\mathbf{t}(\sigma))$ and $-l(\mathbf{t}(\sigma)) \leq s \leq 0$ along $C_-(\mathbf{t}(\sigma))$. With $\mathbf{x} = (x, y)$ and $\tau(\mathbf{x}) = (\tau_1(\mathbf{x}), \tau_2(\mathbf{x}))$ it is shown in [5, (2.4)], that the determinant of the Jacobian

$$(4.2) \quad J := \left| \frac{\partial(x, y)}{\partial(\sigma, s)} \right| = \frac{1}{|\nabla \tau(\sigma, s)|} := \frac{1}{[|\nabla \tau_1(\mathbf{x})|^2 + |\nabla \tau_2(\mathbf{x})|^2]^{1/2}}.$$

For a measurable subset Γ_0 of Γ and $\Omega_0 := \tau^{-1}(\Gamma_0)$, it then follows that, for any $f \in L^2(\Omega_0)$ with $f = 0$ outside Ω_0 ,

$$(4.3) \quad \int_{\Omega_0} f(\mathbf{x}) d\mathbf{x} = \sum_{j \in \mathbb{N}} \int_{e_j} d\sigma \int_{-l(\sigma)}^{l(\sigma)} f(\sigma, s) \frac{1}{|\nabla \tau(\sigma, s)|} ds.$$

This implies, in particular, that for $f = F \circ \tau$ with $F \in L^2(\Gamma_0)$,

$$\int_{\Omega_0} F \circ \tau(\mathbf{x}) d\mathbf{x} = \sum_{j \in \mathbb{N}} \int_{e_j} F(\sigma) d\sigma \int_{-l(\sigma)}^{l(\sigma)} \frac{1}{|\nabla \tau(\sigma, s)|} ds = \int_{\Gamma_0} F(\sigma) \alpha(\sigma) d\sigma,$$

where

$$(4.4) \quad \alpha(\sigma) := \int_{-l(\sigma)}^{l(\sigma)} \frac{1}{|\nabla \tau(\sigma, s)|} ds.$$

The integral

$$(4.5) \quad \beta(\sigma) := \int_{-l(\sigma)}^{l(\sigma)} |\nabla \tau(\sigma, s)| ds$$

will also feature in certain specific regions of Ω .

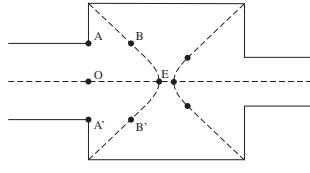
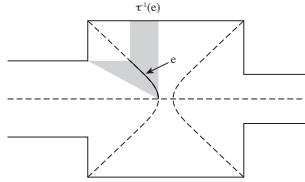


FIGURE 5. The skeleton.

FIGURE
6. $\tau^{-1}(e)$ for
a parabolic edge.

We shall be considering a general R&P domain hereafter, and not the special case of Sections 2 and 3. Therefore we allow for the possibility that the Neumann Laplacian does not have a discrete spectrum. Our first task is to make explicit the change of co-ordinates (4.1) in each region $\tau^{-1}(e)$, $e \in \Gamma$ and then determine the map τ . The edges fall into 3 groups which have to be handled separately. In what follows below, we denote the height of a room by h and of a passage by δ .

Group 1 This consists of edges which are either in a passage or lie in the centre of a room with adjacent parabolic edges. Here, $\sigma = x$, $s = \pm y$, so the determinant of the Jacobian in (4.2) equals 1 and

$$l(\sigma) = \begin{cases} h/2 & \text{in rooms,} \\ \delta/2 & \text{in passages.} \end{cases}$$

Hence,

$$(4.6) \quad \alpha(\sigma) = \beta(\sigma) = \begin{cases} h & \text{in rooms,} \\ \delta & \text{in passages.} \end{cases}$$

Group 2 These are the edges in a room which are straight line segments along the diagonals. In Figure 5 with the origin at O , the edge on the diagonal of the square given by $0 < x < \frac{h-\delta}{2}$ and $y > \delta/2$ lies in this group and the analysis that follows is typical for all edges in this group. First, consider the triangle below the bisecting line, i.e. $\delta/2 < y < h/2 - x$. Here, we re-parameterize points $(x, y) \in \Omega$ by (σ, s) , where σ is the arc length along the skeleton measured from the corner and s is the negative horizontal distance of the point from the skeleton. Thus,

$$(x, y) = \left(\frac{\sigma}{\sqrt{2}} + s, -\frac{\sigma}{\sqrt{2}} + \frac{h}{2} \right), \quad (\sigma, s) = \left(\sqrt{2}\left(\frac{h}{2} - y\right), -\frac{h}{2} + x + y \right).$$

In the triangle above the diagonal, where $\frac{h}{2} - x < y < \frac{h}{2}$, we choose s to be the vertical distance to the skeleton. Here,

$$(x, y) = \left(\frac{\sigma}{\sqrt{2}}, -\frac{\sigma}{\sqrt{2}} + \frac{h}{2} + s \right), \quad (\sigma, s) = \left(\sqrt{2}x, -\frac{h}{2} + x + y \right).$$

We note that in this whole square, we have that

$$\tau(x, y) = \left(\frac{\sigma}{\sqrt{2}}, \frac{h}{2} - \frac{\sigma}{\sqrt{2}} \right),$$

with $0 < \sigma < \frac{h-\delta}{\sqrt{2}}$. Moreover, the determinant of the Jacobian $J = 1/\sqrt{2}$ and $l(\sigma) = \sigma/\sqrt{2}$. Therefore,

$$(4.7) \quad \alpha(\sigma) = \sigma, \quad \beta(\sigma) = 2\sigma.$$

Group 3 These are edges which are such that every point on the edge has a re-entrant corner as one of its two near points. Thus in Figure 5, the parabolic edges BE and the line segment OE belong to this group associated with the re-entrant corner A. We consider the parabolic edge BE. For $y > 0$ this is determined by $|AQ| = |QQ'|$, where $Q = (x_0, y_0)$ is a point on the parabola and $Q' = (x_0, h/2)$. This gives

$$x_0^2 + \left(y_0 - \frac{\delta}{2} \right)^2 = \left(y_0 - \frac{h}{2} \right)^2.$$

After a little algebra this leads to

$$(4.8) \quad y_0 = -\frac{x_0^2}{h-\delta} + \frac{1}{4}(h+\delta).$$

In particular, the parabola intersects the x-axis at $E = \left(\frac{\sqrt{h^2-\delta^2}}{2}, 0 \right)$.

Consider the part of the domain emanating from the re-entrant corner to the parabolic part of the skeleton. Let $\tau(x, y) = (x_0, y_0)$. Then in addition to lying on the parabola, (x_0, y_0) satisfies

$$(4.9) \quad y_0 = \frac{y - \delta/2}{x} x_0 + \frac{\delta}{2}.$$

The arc length along the parabola

$$(4.10) \quad \sigma = \int_{\frac{1}{2}(h-\delta)}^{x_0} \sqrt{1 + \left(\frac{-2x}{h-\delta} \right)^2} dx = \frac{h-\delta}{2} \int_1^{t_0} \sqrt{1+t^2} dt,$$

with $t_0 = \frac{2x_0}{h-\delta}$. A straightforward calculation gives that

$$(4.11) \quad \sigma = \frac{1}{4}(h-\delta) \left(t_0 \sqrt{t_0^2 + 1} + \sinh^{-1}(t_0) - \sqrt{2} - \sinh^{-1}(1) \right).$$

In particular, the length of the parabolic edge is

$$|CE| = \frac{\sqrt{2}}{4} \sqrt{h(h+\delta)} - \frac{\sqrt{2}}{4}(h-\delta) + \frac{h-\delta}{4} \left(\sinh^{-1} \left(\sqrt{\frac{h+\delta}{h-\delta}} \right) - \sinh^{-1}(1) \right).$$

We now use (4.8) and (4.9) to express t_0 in terms of x and y . Eliminating y_0 in (4.8), we get

$$(h-\delta) \left(\frac{y - \delta/2}{x} x_0 + \frac{\delta}{2} - \frac{1}{4}(h+\delta) \right) + x_0^2 = 0.$$

This yields

$$(4.12) \quad t_0^2 + \frac{2y - \delta}{x} t_0 - 1 = 0,$$

so that

$$(4.13) \quad t_0 = \frac{\frac{\delta}{2} - y}{x} + \sqrt{\left(\frac{\frac{\delta}{2} - y}{x}\right)^2 + 1}.$$

For the distance s from (x, y) to (x_0, y_0) this then gives

$$s^2 = \left(x - \frac{h - \delta}{2} t_0\right)^2 + \left(y - \frac{y - \delta/2}{x} \frac{h - \delta}{2} t_0 - \frac{\delta}{2}\right)^2 = \frac{(t_0^2 + 1)^2 (x - \frac{h - \delta}{2} t_0)^2}{4t_0^2}$$

and hence

$$s = -\frac{(t_0^2 + 1)(x - \frac{h - \delta}{2} t_0)}{2t_0}.$$

We need to calculate the determinant of the Jacobian $\frac{\partial(\sigma, s)}{\partial(x, y)}$. Several terms in this Jacobian vanish and we have

$$\left| \frac{\partial(\sigma, s)}{\partial(x, y)} \right| = \left| \frac{\partial\sigma}{\partial t_0} \frac{\partial t_0}{\partial y} \frac{\partial s}{\partial x} \right|.$$

A calculation gives

$$\frac{\partial\sigma}{\partial t_0} = \frac{1}{2} \sqrt{t_0^2 + 1}(h - \delta), \quad \frac{\partial s}{\partial x} = -\frac{t_0^2 + 1}{2t_0}, \quad \frac{\partial t_0}{\partial y} = -\frac{1}{x} - \frac{\delta - 2y}{x\sqrt{(\delta - 2y)^2 + 4x^2}}.$$

Therefore,

$$\left| \frac{\partial(\sigma, s)}{\partial(x, y)} \right| = \frac{(t_0^2 + 1)^{3/2} (h - \delta) \left(\frac{(\delta - 2y)}{\sqrt{(\delta - 2y)^2 + 4x^2}} + 1 \right)}{4t_0 x}.$$

Substitution of t_0 in terms of x and y gives

$$(4.14) \quad \left| \frac{\partial(\sigma, s)}{\partial(x, y)} \right| = \sqrt{2}(h - \delta) \frac{((\delta - 2y)^2 + 4x^2)^{\frac{1}{4}}}{\left(\sqrt{(\delta - 2y)^2 + 4x^2} - \delta + 2y \right)^{3/2}}.$$

To analyse the behaviour of $J^{-1} = \left| \frac{\partial(\sigma, s)}{\partial(x, y)} \right|$ near the re-entrant corner, let

$$x = r \cos \theta, \quad y = \frac{\delta}{2} - r \sin \theta \quad (\text{i.e. } \delta - 2y = 2r \sin \theta).$$

Then

$$\left| \frac{\partial(\sigma, s)}{\partial(x, y)} \right| = \sqrt{2}(h - \delta) \frac{(4r^2)^{\frac{1}{4}}}{(2r - 2r \sin \theta)^{3/2}} = \frac{h - \delta}{\sqrt{2}r(1 - \sin \theta)^{3/2}},$$

so J^{-1} behaves like $1/r$ where r is the distance from the corner. This behaviour of J^{-1} implies that on the parabolic edge

$$(4.15) \quad \alpha(\sigma) < \infty, \text{ but } \beta(\sigma) = \infty.$$

For this reason, we will need to make sure that the weight β does not appear in the analysis on those edges (like the parabolic edges) whose points have re-entrant near points. How we do this will be made apparent in the next section.

5. A STURM-LIOUVILLE OPERATOR

We denote the set of edges of Γ which belong to groups 1 and 2 by Γ_{reg} and those in Group 3 by Γ_{sing} . Note that Γ_{sing} consists of the parabolic edges and ones like the edge OE in Figure 5 which connect an end of a passage and the parabolic edges. The map τ in (4.2) maps a re-entrant corner onto every point on a singular edge, which motivates us to define any function f on $e \in \Gamma_{\text{sing}}$ to be constant.

The underlying Hilbert spaces on Γ are as follows:

$$(5.1) \quad \tilde{L}^2(\Gamma) = \bigoplus_{e \in \Gamma_{\text{reg}}} L^2(e; \alpha(\sigma)d\sigma) \bigoplus_{e \in \Gamma_{\text{sing}}} \mathbb{C}\mathbf{1}_e,$$

where $\mathbf{1}_e$ is the characteristic function of the edge e , $L^2(e; \alpha(\sigma)d\sigma)$ is the weighted Lebesgue space with inner-product

$$(5.2) \quad \int_e f(\sigma) \overline{g(\sigma)} \alpha(\sigma) d\sigma,$$

and with $f = (f_e) \in \tilde{L}^2(\Gamma)$, we have $f_e = \text{constant}$ for $e \in \Gamma_{\text{sing}}$;

$$(5.3) \quad \tilde{H}^1(\Gamma) := \bigoplus_{e \in \Gamma_{\text{reg}}} H^1(e; \alpha, \beta) \bigoplus_{e \in \Gamma_{\text{sing}}} \mathbb{C}\mathbf{1}_e,$$

where for $e \in \Gamma_{\text{reg}}$, $H^1(e; \alpha, \beta)$ is the weighted Sobolev space with inner product

$$(5.4) \quad (f_e, g_e)_{H^1(e; \alpha, \beta)} = \int_e \left\{ \nabla f_e(\sigma) \overline{\nabla g_e(\sigma)} \beta(\sigma) + f_e(\sigma) \overline{g_e(\sigma)} \alpha(\sigma) \right\} d\sigma.$$

The inner-products on $\tilde{L}^2(\Gamma)$ and $\tilde{H}^1(\Gamma)$ are respectively,

$$(f, g) = \sum_{e \in \Gamma} \int_e f(\sigma) \overline{g(\sigma)} \alpha(\sigma) d\sigma$$

and

$$(5.5) \quad \begin{aligned} (f, g)_{\tilde{H}^1(\Gamma)} &= \sum_{e \in \Gamma_{\text{reg}}} \int_e \left([\nabla f(\sigma) \cdot \overline{\nabla g(\sigma)}] \beta(\sigma) + [f(\sigma) \overline{g(\sigma)}] \alpha(\sigma) \right) d\sigma \\ &\quad + \sum_{e \in \Gamma_{\text{sing}}} \int_e [f(\sigma) \overline{g(\sigma)}] \alpha(\sigma) d\sigma. \end{aligned}$$

We define $H_\Gamma + I$ to be the self-adjoint operator in $\tilde{L}^2(\Gamma)$ associated with the $\tilde{H}^1(\Gamma)$ inner-product, where I is the identity operator on $\tilde{L}^2(\Gamma)$. The following theorem is readily proved by a standard argument; cf., [5], Theorem 3.3.

Theorem 5.1. *The domain $D(H_\Gamma)$ of H_Γ consists of sequences $u = (u_e) \in \tilde{H}^1(\Gamma)$ which satisfy the following :*

- (1) $u_e = \text{constant for } e \in \Gamma_{\text{sing}}$;
- (2) *for each $e \in \Gamma_{\text{reg}}$, $\beta u'_e$ is locally absolutely continuous on the interior of e and*

$$(5.6) \quad \lim\{\beta(\sigma) \frac{du_e}{d\sigma}\} = 0$$

as σ tends to the end points of e ;

- (3) *for $e \in \Gamma_{\text{sing}}$, $(H_\Gamma u)_e = 0$;*
- (4) *for $e \in \Gamma_{\text{reg}}$,*

$$(H_\Gamma u)_e(\sigma) = -\frac{1}{\alpha(\sigma)} \frac{d}{d\sigma} \left[\beta(\sigma) \frac{du_e}{d\sigma} \right].$$

An important part in the analysis will be played by the following operator which maps functions on the skeleton to functions on the R&P domain. Define

$$(5.7) \quad T_0 : \tilde{L}^2(\Gamma) \rightarrow L^2(\Omega) \text{ by } T_0 f = f \circ \tau \text{ for } f \in \tilde{L}^2(\Gamma).$$

Lemma 5.2. *Let $e \in \Gamma$. Then for $g \in L^2(\tau^{-1}(e))$ and $t(\sigma) \in e$ we have*

$$(5.8) \quad (T_0^* g)(\sigma) = \frac{1}{\alpha(\sigma)} \int_{-l(\sigma)}^{l(\sigma)} g(\sigma, s) \frac{1}{|\nabla \tau(\sigma, s)|} ds.$$

Proof. For $F \in L^2(e)$,

$$\begin{aligned} \int_e (T_0^* g)(\sigma) \overline{F(\sigma)} d\sigma &= \int_{\tau^{-1}(e)} g(\sigma) \overline{(T_0 F)(\sigma)} d\sigma \\ &= \int_e \overline{F(\sigma)} \left(\frac{1}{\alpha(\sigma)} \int_{-l(\sigma)}^{l(\sigma)} \frac{g(\sigma, s)}{|\nabla \tau(\sigma, s)|} ds \right) \alpha(\sigma) d\sigma, \end{aligned}$$

proving the result. \square

Remark 5.3. Note that in particular, for $t(\sigma) \in e \in \Gamma_{\text{sing}}$, the function $(T_0^* g)(\sigma)$ is constant and takes the value

$$(5.9) \quad (T_0^* g)(\sigma) = \frac{1}{|\tau^{-1}(e)|} \int_{\tau^{-1}(e)} g(x) dx.$$

Lemma 5.4. *T_0 is an isometry and so $T_0^* T_0 = I$, the identity on $\tilde{L}^2(\Gamma)$.*

Proof. Let $F \in \tilde{L}^2(\Gamma)$. Then for any edge e , we have

$$\begin{aligned} \int_{\tau^{-1}(e)} |(F \circ \tau)(x)|^2 dx &= \int_e |F(\sigma)|^2 \int_{-l(\sigma)}^{l(\sigma)} \frac{1}{|\nabla \tau(\sigma, s)|} ds d\sigma \\ (5.10) \quad &= \int_e |F(\sigma)|^2 \alpha(\sigma) d\sigma. \end{aligned}$$

Adding the contributions from all edges shows that $\|T_0 F\|_{L^2(\Omega)} = \|F\|_{\tilde{L}^2(\Gamma)}$ which completes the proof. \square

The proof of the next lemma on how T_0 interacts with derivatives is the same as that of [5, Lemma 3.2].

Lemma 5.5. *For $F \in \tilde{H}^1(\Gamma)$,*

$$(5.11) \quad \sum_{e \in \Gamma} \int_{\tau^{-1}(e)} |\nabla(F \circ \tau)(x)|^2 dx = \sum_{e \in \Gamma_{\text{reg}}} \int_e |F'(\sigma)|^2 \beta(\sigma) d\sigma.$$

Let $\hat{H}^1(\Omega) := \bigoplus_{e \in \Gamma} H^1(\tau^{-1}(e))$.

Corollary 5.6. *The map $T_1 : \tilde{H}^1(\Gamma) \rightarrow \hat{H}^1(\Omega)$ given by $T_1 f = f \circ \tau$ for $f \in \tilde{H}^1(\Gamma)$ is an isometry.*

In the final theorem, E_1 denotes the natural embedding of $\hat{H}^1(\Omega)$ into the space $\bigoplus_{e \in \Gamma} L^2(\tau^{-1}(e))$, and H_Ω denotes the selfadjoint operator associated with the $\hat{H}^1(\Omega)$ inner-product; thus H_Ω is the orthogonal sum

$$H_\Omega = \bigoplus_{e \in \Gamma} (-\Delta_{N, \tau^{-1}(e)}).$$

Theorem 5.7. (1) *The operator $(H_\Gamma + I)^{-1}$ is not compact on $\tilde{L}^2(\Gamma)$ and $0 \in \sigma_{ess}(H_\Gamma)$.*
(2) *E_1 is not compact and $0 \in \sigma_{ess}(H_\Omega)$.*
(3) *If $E_1(T_1 T_1^* - I) : \hat{H}^1(\Omega) \rightarrow L^2(\Omega)$ is compact, then $\sigma_{ess}(H_\Omega) \subseteq \sigma_{ess}(H_\Gamma)$.*

Proof. (1) On the singular edges $(H_\Gamma + I)^{-1}$ is just I . As there are infinitely many singular edges, $(H_\Gamma + I)^{-1}$ is not compact. Moreover, by considering functions supported on one singular edge, we see that 0 is an eigenvalue of H_Γ of infinite multiplicity.
(2) Take an infinite sequence of edges $(e_n)_{n \in \mathbb{N}}$ and consider the sequence of normalised characteristic functions

$$\varphi_n(\mathbf{x}) = \frac{1}{\sqrt{|\tau^{-1}(e_n)|}} \chi_{\tau^{-1}(e_n)}(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega.$$

This bounded sequence in $\hat{H}^1(\Omega)$ has no convergent subsequence in $L^2(\Omega)$. Moreover, every φ_n is an eigenfunction of H_Ω with eigenvalue 0.

(3) The proof of this is similar to that of Theorem 4.4 in [5]. □

It is interesting to compare the results we get in Theorem 5.7 with those obtained in [2] for horn shaped domains where the authors are able to relate the essential spectrum of the Neumann Laplacian on the horn to that of a Schrödinger operator on the skeleton. In our case the re-entrant corners of the Rooms and Passages domain force us to introduce singular edges which cause the operator on the skeleton to decouple and prevent such a detailed result from being obtained.

REFERENCES

- [1] Burenkov, V., Extension of functions preserving certain smoothness and compactness of embeddings for spaces of differentiable functions, *Proc. Steklov Inst. of Math.* **248** (2005), 1-12.
- [2] Davies, E.B. and Simon, B., Spectral properties of the Neumann Laplacian on horns, *Geom. Funct. Anal.* **2** (1992), 105-117.
- [3] Evans, W.D. and Harris, D.J., Sobolev embeddings for generalized ridged domains, *Proc. London Math. Soc.* **54** (1987), 141-175.
- [4] Evans, W.D. and Harris, D.J., Fractals, trees and the Neumann Laplacian, *Math. Ann.* **296** (1993), 493-527.
- [5] Evans, W.D. and Saito, Y., Neumann Laplacians on domains and operators on associated trees, *Quart.J.Math.* **51** (2000), 313-342.
- [6] Filonov, N., On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator (Russian), *Algebra i Analiz* **16** (2004), 172-176. Translation in *St. Petersburg Math. J.* **16** (2005), 413-416.
- [7] Hempel, R., Seco, L.A. and Simon, B., The essential spectrum of Neumann Laplacians on some bounded singular domains, *Journ. Func. Anal.* **102** (1991), 448-483.
- [8] Huxley, M.N., Exponential sums and lattice points III, *Proc. Londion Math. Soc.* **87** (2003), 591-609.
- [9] Netrusov, Y. and Safarov, Y., Weyl asymptotic formula for the Laplacian on domains with rough boundaries, *Commun.Math.Phys.* **253** (2005), 481-509.

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